

# Equivariant Holomorphic Morse Inequalities I: A Heat Kernel Proof

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**Abstract.** Assume that the circle group acts holomorphically on a compact Kähler manifold with isolated fixed points and that the action can be lifted holomorphically to a holomorphic vector bundle. We give a heat kernel proof of the equivariant holomorphic Morse inequalities. We use some techniques developed by Bismut and Lebeau. These inequalities, first obtained by Witten using a different argument, produce bounds on the multiplicities of weights occurring in the twisted Dolbeault cohomologies in terms of the data of the fixed points.

## 1. Introduction

Morse theory obtains topological information of manifolds from the critical points of the functions. Let  $h$  be a Morse function on a compact manifold of real dimension  $n$  and suppose that  $h$  has isolated critical points only. Let  $m_k$  ( $0 \leq k \leq n$ ) be the  $k$ -th Morse number, the number of critical points of Morse index  $k$ . The Lefschetz fixed-point formula says that the alternating sum of  $m_k$  is equal to that of the Betti numbers  $b_k$ :

$$\sum_{k=0}^n (-1)^k m_k = \sum_{k=0}^n (-1)^k b_k. \quad (1.1)$$

Replacing  $(-1)$  by  $t$ , we get two polynomials (Morse and Poincaré polynomials, respectively) in  $t$  that are equal at  $t = -1$ , i.e.,

$$\sum_{k=0}^n m_k t^k = \sum_{k=0}^n b_k t^k + (1+t)q(t) \quad (1.2)$$

for some polynomial  $q(t) = \sum_{k=0}^n q_k t^k$ . The (strong) Morse inequalities assert that  $q(t) \geq 0$  in the sense that  $q_k \geq 0$  for every  $0 \leq k \leq n$ .

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In a celebrated paper [13], Witten showed that the cohomology groups of the de Rham complex  $(\Omega^*, d)$  can be viewed as the space of ground states of a supersymmetric quantum system and that the Morse inequalities can be derived by using a deformation

$$d_h = e^{-h} d e^h \quad (1.3)$$

which preserves the supersymmetry. The idea was used by Bismut [2] to give a heat kernel proof of the Morse inequalities. Let  $d^*$  and  $d_h^*$  be the (formal) adjoints of  $d$  and  $d_h$  (with a choice of Riemannian metrics), respectively, and let

$$\Delta = \{d, d^*\} \quad \text{and} \quad \Delta_h = \{d_h, d_h^*\} \quad (1.4)$$

be the corresponding Laplacians. (We adopt the standard notations of operator (anti-)commutators  $\{A, B\} = AB + BA$  and  $[A, B] = AB - BA$ .) By Hodge theory,

$$\sum_{k=0}^n (-1)^k \text{Tr}_{\Omega^k} \exp(-u^2 \Delta) = \sum_{k=0}^n (-1)^k b_k \quad (1.5)$$

for any  $u > 0$ ; this is in fact the starting point of the heat kernel proof of the index theorem. Similarly, after replacing  $(-1)$  by  $t$ , we obtain

$$\sum_{k=0}^n t^k \text{Tr}_{\Omega^k} \exp(-u^2 \Delta) = \sum_{k=0}^n b_k t^k + (1+t)q_u(t). \quad (1.6)$$

It is a straightforward consequence of Hodge theory that the polynomial  $q_u(t) \geq 0$ . (See for example [2, Theorem 1.3]. A slightly different method is used to show the equivariant version in Lemma 4.1 below.) Since  $(\Omega^*, d_h)$  defines the same cohomology groups as  $(\Omega^*, d)$ , we can replace the heat kernels in (1.6) by those associated to the deformed Laplacian  $\Delta_h$ . It turns out that

$$\lim_{T \rightarrow +\infty} \lim_{u \rightarrow +0} \text{Tr}_{\Omega^k} \exp(-u^2 \Delta_{Th/u^2}) = m_k \quad (0 \leq k \leq n); \quad (1.7)$$

the (strong) Morse inequalities then follow. The heart of the proof is that as  $u \rightarrow 0$  the heat kernel is localized near the critical points of  $h$ , around which the operator consists of  $n$  copies of (supersymmetric) harmonic oscillators whose heat kernels are given by Mehler's formula.

Witten [14] also introduced a holomorphic analog of [13]. Let  $M$  be a compact Kähler manifold of complex dimension  $n$  and let  $E$  be a holomorphic vector bundle over  $M$ . Let  $H^k(M, \mathcal{O}(E))$  be the cohomology groups with coefficients in the sheaf of holomorphic sections of  $E$ , calculated from the twisted Dolbeault complex  $(\Omega^{0,*}(M, E), \bar{\partial}_E)$ . Suppose that the circle group  $S^1$  acts holomorphically and effectively on  $M$  preserving the Kähler structure and that the action can be lifted holomorphically to  $E$ . Then  $e^{\sqrt{-1}\theta}$  also acts on the space of sections by sending a section  $s$  to  $e^{\sqrt{-1}\theta} \circ s \circ e^{-\sqrt{-1}\theta}$ . The induced action on  $\Omega^{0,*}(M, E)$  commutes with the operator  $\bar{\partial}_E$ . Thus we obtain representations of  $S^1$  on  $H^k(M, \mathcal{O}(E))$ ; the multiplicities of weights of  $S^1$  in each cohomology group will be the subject of our investigation. The  $S^1$ -action on  $(M, \omega)$  is clearly symplectic: let  $V$  be the vector field on  $M$  that generates the  $S^1$ -action, then  $L_V \omega = 0$ . If the fixed-point set  $F$  of  $S^1$  on  $M$  is non-empty, then the  $S^1$ -action is Hamiltonian [9], i.e., there is a moment map  $h: M \rightarrow \mathbb{R}$  such

that  $i_V \omega = dh$ . We further assume that  $F$  contains isolated points only. It is well-known that all the Morse indices are even and hence by the lacunary principle,  $h$  is a perfect Morse function:  $m_{2k-1} = b_{2k-1} (= 0)$ , and  $m_{2k} = b_{2k}$  ( $0 \leq k \leq n$ ). However a refined statement is possible because of the complex structure. For each  $p \in F$ ,  $S^1$  acts on  $T_p M$  by the isotropic representation; let  $\lambda_1^p, \dots, \lambda_n^p \in \mathbb{Z} \setminus \{0\}$  be the weights. We define the *orientation index*  $n^p$  of the fixed point  $p \in F$  as the number of weights  $\lambda_k^p < 0$ ; the Morse index of  $h$  at  $p$  is then  $2(n - n_p)$ . (We need to explain our convention in a simple (but non-compact) example  $M = \mathbb{C}$ ,  $\omega = \frac{\sqrt{-1}}{2} dz \wedge d\bar{z}$ , with an  $S^1$ -action of weight  $\lambda \in \mathbb{Z} \setminus \{0\}$ . Since  $V = \sqrt{-1}\lambda(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}})$ , we have  $h = -\frac{1}{2}\lambda|z|^2$ . Also, the weight of the  $S^1$ -action on the function  $z^k$  (a section of the trivial bundle) is  $-k\lambda$  ( $k \in \mathbb{Z}$ ,  $k \geq 0$ ); this leads to a sign convention different from [14] in the main result.) Furthermore,  $S^1$  acts on the fiber  $E_p$  over  $p \in F$ . It is useful to recall a notation in [14]. If the group  $S^1$  has a representation on a finite dimensional complex vector space  $W$ , let  $W(\theta)$  ( $\theta \in \mathbb{R}$ ) be its character. For example, we denote  $E_p(\theta) = \text{tr}_{E_p} e^{\sqrt{-1}\theta}$  and  $H^k(\theta) = \text{tr}_{H^k(M, \mathcal{O}(E))} e^{\sqrt{-1}\theta}$ . The analog of the Lefschetz formula is the fixed-point formula of Atiyah and Bott [1], which we write as an equality of alternating sums [13]:

$$\sum_{p \in F} (-1)^{n_p} E_p(\theta) \prod_{\lambda_k^p > 0} \frac{1}{1 - e^{-\sqrt{-1}\lambda_k^p \theta}} \prod_{\lambda_k^p < 0} \frac{e^{-\sqrt{-1}|\lambda_k^p| \theta}}{1 - e^{-\sqrt{-1}|\lambda_k^p| \theta}} = \sum_{k=0}^n (-1)^k H^k(\theta). \quad (1.8)$$

It turns out that if  $(-1)$  is replaced by  $t$ , the analog of strong Morse inequalities like (1.2) holds. We need the following

**Definition 1.1** Let  $q(\theta) = \sum_{m \in \mathbb{Z}} q_m e^{\sqrt{-1}m\theta} \in \mathbb{R}((e^{\sqrt{-1}\theta}))$  be a formal character of  $S^1$ , we say  $q(\theta) \geq 0$  if  $q_m \geq 0$  for all  $m \in \mathbb{Z}$ ; let  $Q(\theta, t) = \sum_{k=0}^n q_k(\theta) t^k \in \mathbb{R}((e^{\sqrt{-1}\theta}))[t]$  be a polynomial of degree  $n$  with coefficients in  $\mathbb{R}((e^{\sqrt{-1}\theta}))$ , we say  $Q(\theta, t) \geq 0$  if  $q_k(\theta) \geq 0$  for all  $k$ . For two such polynomials  $P(\theta, t)$  and  $Q(\theta, t)$ , we say  $P(\theta, t) \leq Q(\theta, t)$  if  $Q(\theta, t) - P(\theta, t) \geq 0$ .

Using a holomorphic version of supersymmetric quantum mechanics, Witten [14] derived the following

**Theorem 1.2** Suppose  $M$  is a compact Kähler manifold on which  $S^1$  acts holomorphically preserving the Kähler form and with non-empty and discrete fixed-point set and suppose that the  $S^1$ -action can be lifted holomorphically to a holomorphic vector bundle  $E$  over  $M$ . Then under the above assumptions and notations, we have

1. Weak equivariant holomorphic Morse inequalities:

$$H^k(\theta) \leq \sum_{p \in F, n_p = k} E_p(\theta) \prod_{\lambda_k^p > 0} \frac{1}{1 - e^{-\sqrt{-1}\lambda_k^p \theta}} \prod_{\lambda_k^p < 0} \frac{e^{-\sqrt{-1}|\lambda_k^p| \theta}}{1 - e^{-\sqrt{-1}|\lambda_k^p| \theta}}, \quad (1.9)$$

$$H^k(\theta) \leq \sum_{p \in F, n_p = k} E_p(\theta) \prod_{\lambda_k^p > 0} \frac{e^{\sqrt{-1}\lambda_k^p \theta}}{1 - e^{\sqrt{-1}\lambda_k^p \theta}} \prod_{\lambda_k^p < 0} \frac{1}{1 - e^{\sqrt{-1}|\lambda_k^p| \theta}} \quad (0 \leq k \leq n); \quad (1.10)$$

2. Strong equivariant holomorphic Morse inequalities:

$$\sum_{p \in F} t^{n_p} E_p(\theta) \prod_{\lambda_k^p > 0} \frac{1}{1 - e^{-\sqrt{-1}\lambda_k^p \theta}} \prod_{\lambda_k^p < 0} \frac{e^{-\sqrt{-1}|\lambda_k^p| \theta}}{1 - e^{-\sqrt{-1}|\lambda_k^p| \theta}} = \sum_{k=0}^n t^k H^k(\theta) + (1+t)Q^+(\theta, t), \quad (1.11)$$

$$\sum_{p \in F} t^{n-n_p} E_p(\theta) \prod_{\lambda_k^p > 0} \frac{e^{\sqrt{-1}\lambda_k^p \theta}}{1 - e^{\sqrt{-1}\lambda_k^p \theta}} \prod_{\lambda_k^p < 0} \frac{1}{1 - e^{\sqrt{-1}|\lambda_k^p| \theta}} = \sum_{k=0}^n t^k H^k(\theta) + (1+t)Q^-(\theta, t), \quad (1.12)$$

where  $Q^\pm(\theta, t) \geq 0$ ;

3. *Atiyah-Bott fixed-point theorem:*

$$\sum_{p \in F} \frac{E_p(\theta)}{\prod_{k=1}^n (1 - e^{-\sqrt{-1}\lambda_k^p \theta})} = \sum_{k=0}^n (-1)^k H^k(\theta). \quad (1.13)$$

*Proof.* Clearly the weak inequalities (1.9) and (1.10) follow from the strong ones (1.11) and (1.12), respectively. The index formula (1.13) can be recovered by setting  $t = -1$  in either (1.11) or (1.12). Furthermore, we obtain (1.11) from (1.12) after reversing the  $S^1$ -action and replacing  $\theta$  by  $-\theta$ . The whole paper is devoted to a heat kernel proof of (1.12).  $\square$

The cohomology groups  $H^k(M, \mathcal{O}(E))$  as representation spaces of  $S^1$  depend only on the ( $S^1$ -invariant) holomorphic structure on  $E$ . We can choose an  $S^1$ -invariant Hermitian form on  $E$  and let  $d_E = \partial_E + \bar{\partial}_E$  be the unique compatible holomorphic connection. To simplify notations, we drop the subscript  $E$  but keep in mind that

$$\partial^2 = \bar{\partial}^2 = 0 \quad \text{and} \quad d^2 = \{\partial, \bar{\partial}\} = \Omega \wedge \cdot, \quad (1.14)$$

where the curvature  $\Omega$  is a  $(1, 1)$ -form on  $M$  with values in  $\text{End}(E)$ . Let  $d^*, \partial^*, \bar{\partial}^*$  be the (formal) adjoints of  $d, \partial, \bar{\partial}$ , respectively and let

$$\Delta = \{d, d^*\}, \quad \square = \{\partial, \partial^*\}, \quad \bar{\square} = \{\bar{\partial}, \bar{\partial}^*\} \quad (1.15)$$

be the corresponding Laplacians. Following [14], we deform the  $\bar{\partial}$  operator and its Laplacian by

$$\bar{\partial}_h = e^{-h} \bar{\partial} e^h, \quad \bar{\partial}_h^* = e^h \bar{\partial}^* e^{-h}, \quad \bar{\square}_h = \{\bar{\partial}_h, \bar{\partial}_h^*\}. \quad (1.16)$$

The analog of (1.6) holds, where  $b_k$  should be replaced by  $\dim H^k(M, \mathcal{O}(E))$  ( $0 \leq k \leq n$ ). Contrary to the treatment of ordinary Morse theory in [13, 2], the limit of  $\text{Tr}_{\Omega^{0,k}(M, E)} \exp(-u^2 \bar{\square}_{Th/u^2})$  as  $u \rightarrow 0$  does not exist. To see this, we observe that [14] (see also formulas (2.16) and (2.23) below) up to a (bounded) 0-th order operator,  $u^2 \bar{\square}_{Th/u^2}$  is equal to  $\frac{1}{2}u^2 \Delta_{Th/u^2} + \sqrt{-1}T \hat{L}_V$ , where  $\hat{L}_V$  is the infinitesimal action of the circle group  $S^1$ . Since  $\hat{L}_V$  is an (unbounded) first order differential operator, the analysis of [2] that shows localization of heat kernels does not go through. From the physics point of view, the operator  $u^2 \bar{\square}_{Th/u^2}$  near a critical point of  $h$  is the Hamiltonian operator of a (supersymmetric) charged particle in a uniform magnetic field. Therefore the wave functions, and hence the heat kernel, do not localize to any point no matter how strong the magnetic field is. However since  $\hat{L}_V$  commutes with  $u^2 \bar{\square}_{Th/u^2}$ , we can restrict the latter to an eigenspace of the former. The situation changes drastically because  $\hat{L}_V$  is then a constant. Physically, this amounts to fixing the angular momentum with respect to a given point, which does localize the wave functions to that point in the strong field limit. Therefore in this holomorphic setting, we are naturally lead to consider the equivariant heat kernel and consequently, equivariant Morse-type inequalities.

The inequalities due to Demailly [8] have also been referred to in the literature as holomorphic Morse inequalities. The important difference with our case is that Demailly's inequalities do not require a group action and are asymptotic inequalities, as the tensor power of a holomorphic line bundle gets large, whereas the inequalities which we consider are for a fixed holomorphic vector bundle with a holomorphic  $S^1$ -action, and are not merely asymptotic.

In section 2, we study various deformations of the Laplacians on Kähler manifolds. In particular, the operator  $\bar{\square}_h$  is calculated explicitly. We also compare two other deformations  $\bar{\square}_v$  and  $\bar{\square}_{\sqrt{-1}v}$ , which are used in studying complex immersions [5] and holomorphic equivariant cohomology groups [11]. Roughly speaking, the operators  $\frac{1}{2}\Delta_h$ ,  $\bar{\square}_v$  and  $\bar{\square}_{\sqrt{-1}v}$  form a triplet of a certain  $SU(2)$  group. In section 3, we use the technique of [5] to show that as  $u \rightarrow 0$ , the smooth heat kernel associated to the operator  $\exp(-u^2 \bar{\square}_{Th/u^2} + \sqrt{-1}T\hat{L}_V)$  ( $u > 0$ ,  $T > 0$ ) is localized near the fixed-point set  $F$ , and when  $F$  is discrete, the equivariant heat kernel can be approximated by the using the operators with coefficients frozen at the fixed points. The result of the previous section is used to relate by a unitary conjugation the operator  $-u^2 \bar{\square}_{Th/u^2} + \sqrt{-1}T\hat{L}_V$  to  $-u^2 \bar{\square}_{Th/u^2}$  that appears in [5] (but restricted to a certain subspace) plus a 0-th order operator  $-\sqrt{-1}Tr_V$  (as  $u \rightarrow 0$ ) whose action does not depend on the differential forms. This has enabled us to follow the analysis of [5] closely, though a more direct approach without using the conjugation also seems possible. In section 4, we calculate the equivariant heat kernel of the linearized problem using Mehler's formula and then deduce the (strong) equivariant holomorphic Morse inequalities (1.12) by taking the limit  $T \rightarrow +\infty$ . Unlike the argument using small eigenvalues [14], the 0-th order operator  $r_V$  plays a crucial role in the heat kernel calculation.

In a separate paper [15], equivariant holomorphic Morse inequalities with torus and non-Abelian group actions are established and are applied to toric and flag manifolds. The situations with non-isolated fixed points are left for further investigation.

## 2. Deformed Laplacians on Kähler manifolds

Recall that  $E$  is a holomorphic Hermitian vector bundle over a compact Kähler manifold  $(M, \omega)$ . (The Hermitian structure is needed in the proof but not in the statement of Theorem 1.2.) Let  $A_+ = \omega \wedge \cdot$  be the exterior multiplication of  $\omega$  on  $\Omega^{*,*}(M, E)$  and  $A_- = A_+^*$ , its adjoint. Then  $A_3 = \frac{1}{2}[A_+, A_-]$  preserves the bi-grading of  $\Omega^{*,*}(M, E)$ . In fact, the action of  $A_3$  on  $\Omega^{p,q}(M, E)$  is  $\frac{1}{2}(p+q-n)$ , hence  $[A_3, A_\pm] = \pm A_\pm$ . Set  $A_1 = \frac{1}{2}(A_+ + A_-)$  and  $A_2 = -\frac{\sqrt{-1}}{2}(A_+ - A_-)$ , then  $A_a$  ( $a = 1, 2, 3$ ) satisfy the standard  $\mathfrak{su}(2)$  commutation relations

$$[A_a, A_b] = \sqrt{-1}\epsilon_{abc}A_c. \quad (2.1)$$

(See for example [10].) So there is a unitary representation of  $SU(2)$  on  $\Omega^{*,*}(M, E)$ ; let  $S_a(\alpha) = e^{\sqrt{-1}\alpha A_a}$  be the corresponding group elements. We now introduce a slightly more generalized setup.

**Definition 2.1** *Let  $\sigma \in \Omega^{1,1}(M, E)$  be a real-valued  $(1,1)$ -form. Set  $A_+(\sigma) = \sigma \wedge \cdot$ ,  $A_-(\sigma) = A_+^*$ ,  $A_1(\sigma) = \frac{1}{2}(A_+(\sigma) + A_-(\sigma))$ ,  $A_2(\sigma) = -\frac{\sqrt{-1}}{2}(A_+(\sigma) - A_-(\sigma))$  and  $A_3(\sigma) = \frac{1}{2}[A_+, A_-(\sigma)] (= -\frac{1}{2}[A_-, A_+(\sigma)])$ .*

**Remark 2.2** In computations, it is sometimes convenient to introduce local complex coordinates  $\{z^k, k = 1, \dots, n\}$  on  $M$ . The Kähler form  $\omega = \omega_{k\bar{l}} dz^k \wedge d\bar{z}^{\bar{l}}$  is related to the metric  $g = g_{k\bar{l}} dz^k \otimes d\bar{z}^{\bar{l}}$  by  $\omega_{k\bar{l}} = \sqrt{-1} g_{k\bar{l}} = -\omega_{\bar{l}k}$ . Let  $e^k, e^{\bar{l}}$  be the multiplications by  $dz^k, d\bar{z}^{\bar{l}}$ , and  $i_k, i_{\bar{l}}$ , the contractions by  $\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \bar{z}^{\bar{l}}}$ , respectively. Clearly they satisfy the following anti-commutation relations:  $\{e^k, i_l\} = \delta_l^k, \{e^{\bar{k}}, i_{\bar{l}}\} = \delta_{\bar{l}}^{\bar{k}}$  and others = 0. If a  $(1, 1)$ -form  $\sigma = \sigma_{k\bar{l}} dz^k \wedge d\bar{z}^{\bar{l}}$  is real-valued, then all the  $\sigma_{k\bar{l}}$ 's are purely imaginary. Setting  $i^k = g^{k\bar{l}} i_{\bar{l}}$  and  $i^{\bar{l}} = g^{k\bar{l}} i_k$ , we have  $\Lambda_+(\sigma) = \sigma_{k\bar{l}} e^k e^{\bar{l}}, \Lambda_-(\sigma) = -\sigma_{k\bar{l}} i^k i^{\bar{l}}, \Lambda_1(\sigma) = \frac{1}{2} \sigma_{k\bar{l}} (e^k e^{\bar{l}} - i^k i^{\bar{l}}), \Lambda_2(\sigma) = -\frac{\sqrt{-1}}{2} \sigma_{k\bar{l}} (e^k e^{\bar{l}} + i^k i^{\bar{l}})$ , and  $\Lambda_3(\sigma) = -\frac{\sqrt{-1}}{4} \sigma_{k\bar{l}} ([e^k, i^{\bar{l}}] + [e^{\bar{l}}, i^k])$ .

**Lemma 2.3**

$$S_1(\alpha)^{-1} \Lambda_3(\sigma) S_1(\alpha) = \cos \alpha \Lambda_3(\sigma) - \sin \alpha \Lambda_2(\sigma), \quad (2.2)$$

$$S_2(\alpha)^{-1} \Lambda_3(\sigma) S_2(\alpha) = \cos \alpha \Lambda_3(\sigma) + \sin \alpha \Lambda_1(\sigma), \quad (2.3)$$

$$S_3(\alpha)^{-1} \Lambda_3(\sigma) S_3(\alpha) = \Lambda_3(\sigma). \quad (2.4)$$

*Proof.* A straight-forward calculation using the above anti-commutation relations shows that  $[\Lambda_a, \Lambda_b(\sigma)] = \sqrt{-1} \epsilon_{abc} \Lambda_c(\sigma)$ . This means that  $\{\Lambda_a(\sigma)\}$  is an  $SU(2)$  triplet. Hence the result.  $\square$

It is clear that the Hodge relations (see for example [10])

$$[\Lambda_-, \partial] = \sqrt{-1} \bar{\partial}^*, \quad [\Lambda_-, \bar{\partial}] = -\sqrt{-1} \partial^* \quad (2.5)$$

$$[\Lambda_+, \partial^*] = \sqrt{-1} \bar{\partial}, \quad [\Lambda_+, \bar{\partial}^*] = -\sqrt{-1} \partial \quad (2.6)$$

still hold after coupling to the vector bundle  $E$ . Moreover, we have the Bochner-Kodaira-Nakano identities

$$\Delta = \square + \bar{\square}, \quad \bar{\square} - \square = 2\Lambda_3(\sqrt{-1}\Omega), \quad (2.7)$$

which are consequences of (2.5), (2.6) and the graded Jacobi identities. (Since  $E$  is a holomorphic Hermitian bundle,  $\sqrt{-1}\Omega$  is a  $(1, 1)$ -form valued in the subset of  $\text{End}(E)$  which consists of self-adjoint endomorphisms.) These results have been generalized to non-Kähler situations by [7]. When  $E$  is a flat bundle, we recover the usual relation  $\square = \bar{\square} = \frac{1}{2}\Delta$ .

**Lemma 2.4**

$$S_1(\alpha)^{-1} \bar{\square} S_1(\alpha) = \bar{\square} - (1 - \cos \alpha) \Lambda_3(\sqrt{-1}\Omega) - \sin \alpha \Lambda_2(\sqrt{-1}\Omega), \quad (2.8)$$

$$S_2(\alpha)^{-1} \bar{\square} S_2(\alpha) = \bar{\square} - (1 - \cos \alpha) \Lambda_3(\sqrt{-1}\Omega) + \sin \alpha \Lambda_1(\sqrt{-1}\Omega), \quad (2.9)$$

$$S_3(\alpha)^{-1} \bar{\square} S_3(\alpha) = \bar{\square}. \quad (2.10)$$

*Proof.* From (2.5) and (2.6) we deduce that  $S_1(\alpha)^{-1} \bar{\partial} S_1(\alpha) = \cos \frac{\alpha}{2} \bar{\partial} - \sin \frac{\alpha}{2} \partial^*$ . Therefore

$$\begin{aligned} S_1(\alpha)^{-1} \bar{\square} S_1(\alpha) &= \left\{ \cos \frac{\alpha}{2} \bar{\partial} - \sin \frac{\alpha}{2} \partial^*, \cos \frac{\alpha}{2} \bar{\partial}^* - \sin \frac{\alpha}{2} \partial \right\} \\ &= \cos^2 \frac{\alpha}{2} \bar{\square} + \sin^2 \frac{\alpha}{2} \square - \cos \frac{\alpha}{2} \sin \frac{\alpha}{2} (\{\partial, \bar{\partial}\} + \{\partial^*, \bar{\partial}^*\}) \\ &= \bar{\square} - (1 - \cos \alpha) \Lambda_3(\sqrt{-1}\Omega) - \sin \alpha \Lambda_2(\sqrt{-1}\Omega). \end{aligned} \quad (2.11)$$

The second formula follows in the same fashion from  $S_2(\alpha)^{-1}\bar{\partial}S_2(\alpha) = \cos \frac{\alpha}{2}\bar{\partial} - \sqrt{-1}\sin \frac{\alpha}{2}\partial^*$ . The last one is because  $\bar{\square}$  preserves the bi-grading.  $\square$

We now equip  $M$  with a holomorphic  $S^1$ -action which preserves the Kähler structure, hence both the complex structure  $J$  and the Riemannian metric  $g$ . The holomorphic condition  $L_V J = 0$  and the Killing equation  $L_V g = 0$  read, in components,

$$V_{k;l} = V_{\bar{k};\bar{l}} \quad \text{and} \quad V_{k,\bar{l}} + V_{\bar{l},k} = 0, \quad (2.12)$$

respectively. As explained in section 1, we assume that the  $S^1$  fixed-point set  $F$  is non-empty. In this case, there is a moment map  $h: M \rightarrow \mathbb{R}$  satisfying  $i_V \omega = dh$ , or  $h_{,k} = -\sqrt{-1}V_k$  and  $h_{,\bar{k}} = \sqrt{-1}V_{\bar{k}}$ . The equations in (2.12) are equivalent to

$$h_{,k;l} = h_{,\bar{k};\bar{l}} = 0 \quad \text{and} \quad h_{,k,\bar{l}} = h_{,\bar{l},k}. \quad (2.13)$$

(The second part is of course the symmetry of the Hessian.) Also notice the real-valued  $(1,1)$ -form

$$dJdh = di_V g = -2\sqrt{-1}h_{,k;\bar{l}}dz^k \wedge d\bar{z}^{\bar{l}}. \quad (2.14)$$

We further assume that the  $S^1$ -action can be lifted holomorphically to the bundle  $E$ . We can choose an  $S^1$ -invariant Hermitian form on  $E$ . Then the connection  $d = d_E$  is also  $S^1$ -invariant. The group element  $e^{\sqrt{-1}\theta} \in S^1$  acts on a section  $s$  by  $s \mapsto e^{\sqrt{-1}\theta} \circ s \circ e^{-\sqrt{-1}\theta}$ . Let  $\hat{L}_V$  be the infinitesimal generator of this  $S^1$ -action on  $\Omega^{*,*}(M, E)$  and let  $L_V = \{i_V, d\}$  be the standard Lie derivative. Then the operator

$$r_V = \hat{L}_V + L_V \quad (2.15)$$

is an element of  $\Gamma(M, \text{End}(E))$ . Over the fiber of a fixed point  $p \in F$ ,  $r_V(p)$  is simply the representation of  $\text{Lie}(S^1)$  on  $E_p$ ; this is independent of the choice of the connections on  $E$ .

**Remark 2.5** 1.  $\bar{\square}_h$  commutes with the  $S^1$ -action. Since the connection, the complex structure, and the moment map  $h$  are all  $S^1$ -invariant, we get  $[\hat{L}_V, d] = 0$ ,  $[\hat{L}_V, \bar{\partial}] = 0$  and  $[\hat{L}_V, \bar{\partial}_h] = 0$ . Taking the adjoint, we get  $[\hat{L}_V, \bar{\partial}_h^*] = 0$  and  $[\hat{L}_V, \bar{\square}_h] = 0$ .

2.  $\bar{\square}_h$  also commutes with a  $U(1)$  subgroup of  $SU(2)$ . Since  $\bar{\square}_h$  preserves the bi-grading,  $[A_3, \bar{\square}_h] = 0$ , hence  $S_3(\alpha)^{-1} \bar{\square}_h S_3(\alpha) = \bar{\square}_h$ .

### Proposition 2.6

$$\bar{\square}_h = \bar{\square} + \frac{1}{2}|dh|^2 - A_3(dJdh) - \sqrt{-1}r_V + \sqrt{-1}\hat{L}_V. \quad (2.16)$$

*Proof.* Let  $D_k, D_{\bar{l}}$  be the covariant derivative along  $\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \bar{z}^{\bar{l}}}$ , respectively. Then  $\bar{\partial} = e^{\bar{l}}D_{\bar{l}}, \bar{\partial}^* = -i^k D_k$  and  $\bar{\partial}_h = \bar{\partial} + e^{\bar{l}}h_{,\bar{l}}, \bar{\partial}_h^* = \bar{\partial}^* + i^k h_{,k}$ . So

$$\begin{aligned} \bar{\square}_h &= \{\bar{\partial}, \bar{\partial}^*\} + \{e^{\bar{l}}, i^k\}h_{,\bar{l}}h_{,k} + \{\bar{\partial}, i^k h_{,k}\} + \{\bar{\partial}^*, e^{\bar{l}}h_{,\bar{l}}\} \\ &= \bar{\square} + g^{k\bar{l}}h_{,k}h_{,\bar{l}} + (h_{,k,\bar{l}}e^{\bar{l}}i^k + h_{,k}D^{\bar{k}}) - (h_{,\bar{l},k}i^k e^{\bar{l}} + h_{,\bar{l}}D^{\bar{k}}) \\ &= \bar{\square} + \frac{1}{2}|dh|^2 - A_3(dJdh) + h_{,k,\bar{l}}(e^{\bar{l}}i^k - e^k i^{\bar{l}}) - \sqrt{-1}(V_k D^k + V_{\bar{l}} D^{\bar{l}}). \end{aligned} \quad (2.17)$$

On the other hand,

$$\begin{aligned}
L_V &= \{\partial + \bar{\partial}, V_k i^k + V_{\bar{l}} i^{\bar{l}}\} \\
&= V_k D^k + V_{\bar{l}} D^{\bar{l}} + V_{k,\bar{l}} e^{\bar{l}} i^k + V_{\bar{l},k} e^k i^{\bar{l}} \\
&= V_k D^k + V_{\bar{l}} D^{\bar{l}} + \sqrt{-1} h_{k,\bar{l}} (e^{\bar{l}} i^k - e^k i^{\bar{l}}).
\end{aligned} \tag{2.18}$$

(2.16) follows from (2.17), (2.18) and (2.15).  $\square$

We also define two different deformations. Let  $v = V^{1,0}$  be the holomorphic component of  $V$ . Set

$$\bar{\partial}_v = \bar{\partial} + i_v, \quad \bar{\square}_v = \{\bar{\partial}_v, \bar{\partial}_v^*\} \tag{2.19}$$

and

$$\bar{\partial}_{\sqrt{-1}v} = \partial + \sqrt{-1}i_v, \quad \bar{\square}_{\sqrt{-1}v} = \{\bar{\partial}_{\sqrt{-1}v}, \bar{\partial}_{\sqrt{-1}v}^*\}. \tag{2.20}$$

Then straightforward calculations similar to what leads to (2.16) yield

$$\bar{\square}_v = \bar{\square} + \frac{1}{2}|dh|^2 + A_1(dJdh) \tag{2.21}$$

and

$$\bar{\square}_{\sqrt{-1}v} = \bar{\square} + \frac{1}{2}|dh|^2 + A_2(dJdh). \tag{2.22}$$

It is also interesting to compare the deformation  $\Delta_h$  in [13] of the usual Laplacian (coupled to the bundle  $E$ ).

Using (2.13) again, we get

$$\frac{1}{2}\Delta_h = \frac{1}{2}\Delta + \frac{1}{2}|dh|^2 - A_3(dJdh). \tag{2.23}$$

When the bundle  $E$  is flat, the only difference of  $\bar{\square}_v$ ,  $\bar{\square}_{\sqrt{-1}v}$  and  $\frac{1}{2}\Delta_h$  are in the terms  $A_a(dJdh)$  ( $a = 1, 2, 3$ ). In this case, deformations break the  $SU(2)$  symmetry of  $\bar{\square}$  to  $U(1)$ , while the  $\frac{\pi}{2}$  rotations in  $SU(2)$  interchanges the three operators  $\bar{\square}_v$ ,  $\bar{\square}_{\sqrt{-1}v}$  and  $\frac{1}{2}\Delta_h$ .

Finally we come to the relation of  $\bar{\square}_h$  and  $\bar{\square}_v$ .

### Proposition 2.7

$$S_2(-\frac{\pi}{2})^{-1} \bar{\square}_h S_2(-\frac{\pi}{2}) = \bar{\square}_v - A_3(\sqrt{-1}\Omega) - A_1(\sqrt{-1}\Omega) - \sqrt{-1}r_V + \sqrt{-1}\hat{L}_V. \tag{2.24}$$

*Proof.* Using (2.16), (2.9) and (2.3), we get

$$\begin{aligned}
S_2(\alpha)^{-1} \bar{\square}_h S_2(\alpha) &= \bar{\square} - (1 - \cos \alpha) A_3(\sqrt{-1}\Omega) + \sin \alpha A_1(\sqrt{-1}\Omega) + \frac{1}{2}|dh|^2 \\
&\quad - \cos \alpha A_3(dJdh) - \sin \alpha A_1(dJdh) - \sqrt{-1}r_V + \sqrt{-1}\hat{L}_V.
\end{aligned} \tag{2.25}$$

Set  $\alpha = -\frac{\pi}{2}$ .  $\square$



### 3. Localization to the fixed-point set

**Definition 3.1** For  $u > 0$ ,  $T \geq 0$ , let  $P_{u,T}(x, x')$  ( $x, x' \in M$ ) be the smooth kernel associated to the operator  $\exp(-u^2 \bar{\square}_{Th/u} + \sqrt{-1}uT\hat{L}_V)$  calculated with respect to the Riemannian volume element  $dv_M$  of  $M$ .

So for  $x \in M$ ,  $P_{u,T}(x, x) \in \text{End}(\Omega^{0,*}(M, E))|_x$ . Moreover,  $e^{\theta\hat{L}_V}P_{u,T}(e^{-\sqrt{-1}\theta}x, x) \in \text{End}(\Omega^{0,*}(M, E))|_x$ .

**Proposition 3.2** Take  $\alpha > 0$ . There exist  $c > 0$ ,  $C > 0$  such that for all  $x \in M$  with  $d(x, F) \geq \alpha$ ,  $e^{\sqrt{-1}\theta} \in S^1$ , and all  $u \in (0, 1]$ , we have

$$|P_{u,T/u}(e^{-\sqrt{-1}\theta}x, x)| \leq ce^{-C/u^2}. \quad (3.1)$$

*Proof.* We use the techniques (and the notations) of [5]. Consider  $i: F \rightarrow M$  as an embedding of compact complex manifolds. Let  $\eta = i^*E$  and  $\xi_k = A^k T^{*(1,0)}M \otimes E$  ( $k = 0, \dots, n$ ). Then

$$(\xi, i_v): 0 \rightarrow \xi_n \rightarrow \xi_{n-1} \rightarrow \dots \rightarrow \xi_0 \quad (3.2)$$

is a holomorphic chain complex of vector bundles on  $M$ . Since  $F$  is discrete, (3.2), together with the restriction map  $\xi_0|_F \rightarrow \eta$ , is a resolution of the sheaf  $i_*\mathcal{O}_F(\eta)$ . The elliptic operator considered in [5] is

$$u^2 \bar{\square}_{Tv/u} = (uD^M + T\hat{V})^2 = u^2(D^M)^2 + uT\{D^M, \hat{V}\} + T^2\hat{V}^2 \quad (3.3)$$

acting on  $\Omega^{*,0}(M) \hat{\otimes} \Omega^{0,*}(M, E) = \Omega^{*,*}(M, E)$ , where  $D^M = \bar{\partial}_v + \bar{\partial}_v^*$ ,  $\hat{V} = i_v + i_v^*$ . Particularly important is that the operator  $\{D^M, \hat{V}\}$  is of order zero, hence  $uT\{D^M, \hat{V}\}$  is uniformly bounded for  $u \in (0, 1]$ ,  $T \in [0, 1/u]$ . We now extend the domain of our operator  $\bar{\square}_h$  from (the  $L^2$ -completion of)  $\Omega^{0,*}(M, E)$  to (that of)  $\Omega^{*,*}(M, E)$ . Since the operator preserves the bi-grading of  $\Omega^{*,*}(M, E)$ , it suffices to prove (3.1) for the heat kernel with the extended domain. Using Proposition 2.7, we have

$$S_2(-\frac{\pi}{2})^{-1}(u^2 \bar{\square}_{Th/u} - \sqrt{-1}uT\hat{L}_V)S_2(-\frac{\pi}{2}) = u^2 \bar{\square}_{Tv/u} - r_{u,T}. \quad (3.4)$$

Here  $r_{u,T} = uA_3(\sqrt{-1}\Omega) + uA_1(\sqrt{-1}\Omega) + uT\sqrt{-1}r_V$  is also uniformly bounded for  $u \in (0, 1]$ ,  $T \in [0, 1/u]$ . The operator on the right hand side of (3.4) has the same heat kernel  $P_{u,T}$  up to a conjugation by  $S_2(-\frac{\pi}{2})$ . Therefore the proof of [5, Proposition 11.10] implies that there exist a sufficiently small  $b > 0$  (determined by the injectivity radius of  $M$ ), and  $c_1 > 0$ ,  $C_1 > 0$  such that for all  $x_0 \in M$ ,  $u \in (0, 1]$ ,  $T \in [0, 1/u]$ ,  $x \in B(x_0, b/2)$ , we have

$$|(P_{u,T} - P_{u,T}^{x_0})(x, x)| \leq c_1 e^{-C_1/u^2}. \quad (3.5)$$

Here  $P_{u,T}^{x_0}$  is the smooth heat kernel of the same operator with Dirichlet conditions on  $\partial B^M(x_0, b)$ . Hence

$$|(P_{u,T/u} - P_{u,T/u}^x)(x, x)| \leq c_1 e^{-C_1/u^2} \quad (3.6)$$

for all  $x \in M$  and  $T \geq 0$ . (The condition  $T \leq 1$  can be lifted by a scaling argument.) Since  $\hat{V}$  is invertible on  $M \setminus F$ , by the proof of [5, Proposition 12.1], for any  $\alpha > 0$  there exist  $c_2, C_2, C'_2 > 0$  such that

$$|P_{u,T/u}^x(x, x)| \leq \frac{c_2}{u^{2n}} e^{-C_2 T^2/u^2 + C'_2 T} \quad (3.7)$$

for any  $x \in M$  with  $d(x, F) \geq \alpha$ . (3.6) and (3.7) imply that for some  $c, C > 0$ ,

$$|P_{u,T/u}(x, x)| \leq c e^{-C/u^2}. \quad (3.8)$$

Formula (3.1) follows from [4, equation (12.7)]:

$$|P_{u,T/u}(e^{-\sqrt{-1}\theta}x, x)| \leq |P_{u,T/u}(e^{-\sqrt{-1}\theta}x, e^{-\sqrt{-1}\theta}x)|^{\frac{1}{2}} |P_{u,T/u}(x, x)|^{\frac{1}{2}} \quad (3.9)$$

and from the  $S^1$ -invariance of  $|P_{u,T/u}(x, x)|$ .  $\square$

Clearly, Proposition 3.2 is valid without the assumption that the fixed-point set  $F$  is discrete; in general  $F$  is a symplectic, hence Kähler submanifold of  $M$ . The result can also be proved using the method of [12]. The proof here is similar to that of [6, Theorem 3.11] except that, without the  $\mathbb{Z}_2$  symmetry there, we do not get a vanishing result in Proposition 3.4 below.

**Definition 3.3** *Let  $R_p(\theta)$  be the isotropy representation of  $e^{\sqrt{-1}\theta} \in S^1$  on  $T_p M$  and let  $Z = (z^1, \dots, z^n)$  be the linear complex coordinates on  $T_p M$  such that the action of  $R_p(\theta)$  is*

$$R_p(\theta)(z^1, \dots, z^n) = (e^{\sqrt{-1}\lambda_1^p \theta} z^1, \dots, e^{\sqrt{-1}\lambda_n^p \theta} z^n). \quad (3.10)$$

For  $T \geq 0$ , set

$$\mathcal{B}_{T^2}^{2,p} = \frac{1}{2} \Delta^p + \frac{1}{2} T^2 \sum_{k=1}^n |\lambda_k^p|^2 |z^k|^2 + T \sum_{k=1}^n \sqrt{-1} \lambda_k^p (e^k e^{\bar{k}} - i_k i_{\bar{k}}) \quad (3.11)$$

and

$$\mathcal{C}_{T^2}^{2,p} = \frac{1}{2} \Delta^p + \frac{1}{2} T^2 \sum_{k=1}^n |\lambda_k^p|^2 |z^k|^2 - \frac{1}{2} T \sum_{k=1}^n \lambda_k^p ([e^k, i_k] + [e^{\bar{k}}, i_{\bar{k}}]), \quad (3.12)$$

where  $\Delta^p$  is the (positive) flat Laplacian on  $T_p M$ .

It is easy to see that the  $SU(2)$  group elements  $S_a(\alpha)$  ( $a = 1, 2, 3$ ) act on  $\Omega^{*,*}(T_p M)$  and that

$$S_2(-\frac{\pi}{2})^{-1} \mathcal{C}_{T^2}^{2,p} S_2(-\frac{\pi}{2}) = \mathcal{B}_{T^2}^{2,p}. \quad (3.13)$$

This can be used to recover [3, Theorem 1.6] from [13]. Moreover, if  $Q_{T^2}^p$  is the heat kernel associated to  $\exp(-\mathcal{C}_{T^2}^{2,p})$ , then  $S_2(-\frac{\pi}{2})^{-1} Q_{T^2}^p S_2(-\frac{\pi}{2})$  is that of  $\exp(-\mathcal{B}_{T^2}^{2,p})$ .

**Proposition 3.4** *For  $T > 0$ ,  $\theta \in \mathbb{R}$ ,*

$$\begin{aligned} & \lim_{u \rightarrow 0} \text{Tr}_{\Omega^{0,k}(M,E)} \exp[-u^2 \bar{\square}_{Th/u^2} + (\theta + \sqrt{-1}T) \hat{L}_V] \\ &= \sum_{p \in F} E_p(\theta + \sqrt{-1}T) \text{Tr}_{\Omega^{0,k}(T_p M)} [R_p(\theta) \exp(-\mathcal{C}_{T^2}^{2,p})]. \end{aligned} \quad (3.14)$$

Moreover, the limit is uniform in  $\theta$ .

*Proof.* We recall the notations of [5, §11-12]. Fix a small  $\epsilon > 0$ . For  $p \in F$ , the ball  $B^{T_p M}(0, \epsilon) \subset T_p M$  is identified with the ball  $B^M(0, \epsilon) \subset M$  by the exponential map. Let  $k'(Z) = \det(d_Z \exp)$ ,  $Z \in T_p M$ , be the Jacobian. Then  $dv_{T_p M}(Z) = k'(Z)dv_M(Z)$  and  $k(0) = 1$ . We also identify  $T_Z M$ ,  $E_Z$  with  $T_p M$ ,  $E_p$ , respectively, by the parallel transports along the geodesic connecting  $p$  and  $Z$ . The operators  $D^M$  and  $\hat{V}$  now act on smooth sections of  $\Lambda^*(T_p M) \otimes E_p$  over  $B^{T_p M}(0, \epsilon)$ . The setup here is simpler than that in [5, 4] because  $F$  is discrete and because of the resolution (3.2) we choose. (Using the notations in [5, §8.f], here  $\xi^+ = 0$  and  $\xi^- = \xi$ .) Following [5, §11.h-i and §12.d-e], we define

$$L_{u,T}^{1,p} = u^2(1 - \rho^2(Z))\frac{\Delta^p}{2} + \rho^2(Z)(u^2 \bar{\square}_{T_{v/u}} - r_{u,T}), \quad (3.15)$$

where  $\rho(Z) = \rho(|Z|)$  is a smooth function such that  $\rho(Z) = 1$  if  $|Z| \leq \frac{\epsilon}{4}$  and  $\rho(Z) = 0$  if  $|Z| \geq \frac{\epsilon}{2}$ , and

$$L_{u,T}^{3,p} = F_u^{-1} L_{u,T}^{1,p} F_u, \quad (3.16)$$

where  $F_u$  is a rescaling:  $F_u h(Z) = h(Z/u)$ . Let  $P_{u,T}^{1,p}(Z, Z')$ ,  $P_{u,T}^{3,p}(Z, Z')$  be the smooth heat kernel associated to the operators  $\exp(-L_{u,T}^{1,p})$ ,  $\exp(-L_{u,T}^{3,p})$ , respectively, calculated in the volume element  $dv_{T_p M}$ . Clearly

$$u^{2n} P_{u,T}^{1,p}(uZ, uZ') = P_{u,T}^{3,p}(Z, Z'). \quad (3.17)$$

The only term in  $L_{u,T}^{3,p}$  that did not appear in [5, equation (11.60)] is  $-\rho^2(uZ)r_{u,T}(uZ)$ . It is easy to see that for  $u \in (0, 1]$ ,  $T \in [1, 1/u]$ , the operator  $1_{u|Z| \leq \epsilon/2} r_{u,T}(uZ)$  is uniformly bounded with respect to the norm  $|\cdot|_{u,T,0,0}$  in [5, Definition 11.23]. This, together with [5, Proposition 11.24], is enough to establish the results in [5, Theorem 11.26] (in the special case of  $Z_0 = 0$ ) for  $L_{u,T}^{3,p}$ . We can then proceed as the proof of [5, Theorem 11.31] and obtain the analog of [5, Theorem 12.14] on the uniform estimates of  $P_{u,T/u}^{3,p}$ . In particular, for any  $m \in \mathbb{N}$ , there exists  $c > 0$  such that if  $u \in (0, 1]$ , then

$$|P_{u,T/u}^{3,p}(Z, Z)| \leq \frac{c}{(1 + |Z|)^m} \quad (3.18)$$

for  $|Z| \leq \frac{\epsilon}{8u}$ . Using (3.17) and the analog of (3.9), we get

$$u^{2n} |P_{u,T/u}^{1,p}(uR_p^{-1}(\theta)Z, uZ)| \leq \frac{c}{(1 + |Z|)^m}. \quad (3.19)$$

Next, from (3.15) and (3.16), we get

$$L_{u,T/u}^{3,p} = \frac{1}{2}u^2(1 - \rho^2(uZ))\Delta^p + \rho^2(uZ)(D^M)^2 + \rho^2(uZ)(T\{D^M, \hat{V}\} + u^{-2}T^2\hat{V}^2(uZ) - r_{u,T/u}(uZ)). \quad (3.20)$$

It is easy to see that as  $u \rightarrow 0$ ,  $r_{u,T/u}(uZ) \rightarrow \sqrt{-1}Tr_V(p)$ ; the rest of the terms in (3.20) tends to  $\mathcal{B}_{T^2}^{2,p}$  by [5, Propositions 12.10, 12.12]. Hence

$$L_{u,T/u}^{3,p} \rightarrow \mathcal{B}_{T^2}^{2,p} - \sqrt{-1}Tr_V(p), \quad \text{as } u \rightarrow 0. \quad (3.21)$$

Proceed as the proof of [5, Theorem 12.16] (with the simplification  $L_{u,1} = L_{u,T/u}^{3,p}$  and  $L_{u,2}, L_{u,3}, L_{u,4} = 0$ ) and as [5, §12.i], we conclude that

$$S_2(-\frac{\pi}{2})^{-1} P_{u,T/u}^{3,p} S_2(-\frac{\pi}{2}) \rightarrow S_2(-\frac{\pi}{2})^{-1} Q_{T^2}^p S_2(-\frac{\pi}{2}) \otimes e^{\sqrt{-1}Tr_V(p)}, \quad \text{as } u \rightarrow 0 \quad (3.22)$$

in the sense of distributions on  $T_p M \times T_p M$ . By the uniform estimates on  $P_{u,T/u}^{3,p}$ ,

$$e^{\theta \hat{L}_V} P_{u,T/u}^{3,p}(R^{-1}(\theta)Z, Z) \rightarrow R_p(\theta)Q_{T^2}^p(R_p^{-1}(\theta)Z, Z) \otimes e^{(\theta + \sqrt{-1}T)rv(p)}, \quad \text{as } u \rightarrow 0 \quad (3.23)$$

uniformly in  $\theta$  and in  $Z$  belonging to any compact set in  $T_p M$ . Using (3.17) and taking the (local) trace over anti-holomorphic forms only, we get

$$\begin{aligned} & \lim_{u \rightarrow 0} u^{2n} \operatorname{tr}_{\Omega_p^{0,k} \otimes E_p} [e^{\theta \hat{L}_V} P_{u,T/u}^{1,p}(uR^{-1}(\theta)Z, uZ)] \\ &= E_p(\theta + \sqrt{-1}T) \operatorname{tr}_{\Omega_p^{0,k}} [R_p(\theta)Q_{T^2}^p(R_p^{-1}(\theta)Z, Z)]. \end{aligned} \quad (3.24)$$

The arguments leading to (3.6) imply (see [5, §12.d] and [4, §12.d]) that there are  $c_0, C_0 > 0$  such that for all  $u \in (0, 1]$  and  $Z \in T_p M$  with  $|Z| \leq \frac{\epsilon}{8}$ , we have

$$|P_{u,T/u}((p, R_p^{-1}(\theta)Z), (p, Z))k'(Z) - P_{u,T/u}^{1,p}(R_p^{-1}(\theta)Z, Z)| \leq c_0 e^{-C_0/u^2}. \quad (3.25)$$

Therefore in (3.24) and (3.19),  $P_{u,T/u}^{1,p}(uR_p^{-1}(\theta)Z, uZ)$  can be replaced by  $P_{u,T/u}((p, uR_p^{-1}(\theta)Z), (p, uZ))k'(uZ)$  for  $|Z| \leq \frac{\epsilon}{8u}$ . By the dominated convergence theorem (as in [5, Remark 12.5], but adapted to take into account uniform convergence), we get

$$\begin{aligned} & \lim_{u \rightarrow 0} \int_{B^M(F, \epsilon/8)} \operatorname{tr}_{\Omega_x^{0,k} \otimes E_x} [e^{\theta \hat{L}_V} P_{u,T/u}(e^{-\sqrt{-1}\theta} x, x)] dv_M(x) \\ &= \sum_{p \in F} E_p(\theta + \sqrt{-1}T) \int_{T_p M} \operatorname{tr}_{\Omega_p^{0,k}} [R_p(\theta)Q_{T^2}^p(R_p^{-1}(\theta)Z, Z)] dv_{T_p M}(Z) \end{aligned} \quad (3.26)$$

uniformly in  $\theta$ . By Proposition 3.2, we can replace the domain of the integration on the left hand side by  $M$  and thus the proposition follows.  $\square$

**Definition 3.5** Let  $q(u, \theta) = \sum_{m \in \mathbb{Z}} q_m(u) e^{\sqrt{-1}m\theta}$  be a family of formal characters of  $S^1$  parameterized by  $u \in \mathbb{R}$  and let  $q(\theta) = \sum_{m \in \mathbb{Z}} q_m e^{\sqrt{-1}m\theta} \in \mathbb{R}((e^{\sqrt{-1}\theta}))$ . We say that  $\lim_{u \rightarrow u_0} q(u, \theta) = q(\theta)$  in  $\mathbb{R}((e^{\sqrt{-1}\theta}))$  if for all  $m \in \mathbb{Z}$ ,  $\lim_{u \rightarrow u_0} q_m(u) = q_m$ .

**Corollary 3.6** For  $T > 0$ , the limit (3.14) holds in  $\mathbb{R}((e^{\sqrt{-1}\theta}))$ .

*Proof.* From Proposition 3.4, we know that as  $u \rightarrow 0$ ,

$$\operatorname{Tr}_{\Omega^{0,k}(M,E)} \exp[-u^2 \bar{\square}_{Th/u^2} + (\theta + \sqrt{-1}T)\hat{L}_V] - \sum_{p \in F} E_p(\theta + \sqrt{-1}T) \operatorname{Tr}_{\Omega^{0,k}(T_p M)} [R_p(\theta) \exp(-\mathcal{C}_{T^2}^{2,p})] \rightarrow 0 \quad (3.27)$$

uniformly in  $\theta$ , and hence in  $L^2(S^1)$  as well. This implies that all the Fourier coefficients of the left hand side tend to 0. The result follows.  $\square$

## 4. Proof of the theorem

As explained in the introduction, the heat kernel proof of equivariant Morse-type inequalities is based on the following

**Lemma 4.1** *For  $u > 0$ ,  $T > 0$ , we have*

$$\sum_{k=0}^n t^k \operatorname{Tr}_{\Omega^{0,k}(M,E)} \exp(-u^2 \bar{\square}_{Th/u^2} + \theta \hat{L}_V) = \sum_{k=0}^n t^k H^k(\theta) + (1+t) Q_{u,T}(\theta, t) \quad (4.1)$$

in  $\mathbb{R}((e^{\sqrt{-1}\theta}))[[t]]$  for some  $Q_{u,T}(\theta, t) \geq 0$ .

*Proof.* Recall that  $\bar{\square}_{Th/u^2} = \{\bar{\partial}_{Th/u^2}, \bar{\partial}_{Th/u^2}^*\}$ . Since  $\bar{\partial}_{Th/u^2}$  and  $\bar{\partial}$  differ by an  $S^1$ -invariant conjugation, their cohomologies are isomorphic as representations of  $S^1$ . Using the ( $S^1$ -equivariant) Hodge decomposition, we get

$$\begin{aligned} \operatorname{Tr}_{\Omega^{0,k}(M,E)} \exp(-u^2 \bar{\square}_{Th/u^2} + \theta \hat{L}_V) &= H^k(\theta) + \operatorname{Tr}_{\bar{\partial}_{Th/u^2}^* \Omega^{0,k+1}(M,E)} \exp(-u^2 \bar{\partial}_{Th/u^2}^* \bar{\partial}_{Th/u^2} + \theta \hat{L}_V) \\ &\quad + \operatorname{Tr}_{\bar{\partial}_{Th/u^2} \Omega^{0,k-1}(M,E)} \exp(-u^2 \bar{\partial}_{Th/u^2} \bar{\partial}_{Th/u^2}^* + \theta \hat{L}_V) \end{aligned} \quad (4.2)$$

as formal characters of  $S^1$ . Notice that the spectrum of the operator  $\bar{\partial}_{Th/u^2}^* \bar{\partial}_{Th/u^2}$  on (the closure of)  $\bar{\partial}_{Th/u^2}^* \Omega^{0,k+1}(M,E)$  is identical to that of  $\bar{\partial}_{Th/u^2} \bar{\partial}_{Th/u^2}^*$  on (the closure of)  $\bar{\partial}_{Th/u^2} \Omega^{0,k-1}(M,E)$ . Since the  $S^1$ -action commutes with all the operators, we obtain

$$\begin{aligned} &\operatorname{Tr}_{\bar{\partial}_{Th/u^2}^* \Omega^{0,k+1}(M,E)} \exp(-u^2 \bar{\partial}_{Th/u^2}^* \bar{\partial}_{Th/u^2} + \theta \hat{L}_V) \\ &= \operatorname{Tr}_{\bar{\partial}_{Th/u^2} \Omega^{0,k-1}(M,E)} \exp(-u^2 \bar{\partial}_{Th/u^2} \bar{\partial}_{Th/u^2}^* + \theta \hat{L}_V) \geq 0 \end{aligned} \quad (4.3)$$

in  $\mathbb{R}((e^{\sqrt{-1}\theta}))$ . We denote either of the expressions in (4.3) by  $Q_{u,T}^k(\theta)$ . Summing over  $k = 0, \dots, n$  in (4.2), we obtain (4.1) with  $Q_{u,T}(\theta, t) = \sum_{k=0}^n Q_{u,T}^k(\theta) t^k \geq 0$ .  $\square$

We now take the limit  $u \rightarrow 0$ . To use Proposition 3.4 or Corollary 3.6, we need the following result on the equivariant heat kernel of the anti-holomorphic sector of the (supersymmetric) harmonic oscillator.

**Lemma 4.2** *For  $T > 0$ ,*

$$\operatorname{Tr}_{\Omega^{0,k}(T_p M)} [R_p(\theta) \exp(-\mathcal{C}_{T^2}^{2,p})] = \sum_{I \subset \{1, \dots, n\}, |I|=k} \frac{e^{-T \sum_{k=0}^n |\lambda_k^p| - T \sum_{k \notin I} \lambda_k^p + \sqrt{-1}\theta \sum_{k \in I} \lambda_k^p}}{\prod_{k=1}^n [(1 - e^{-(T - \sqrt{-1}\theta)|\lambda_k^p|})(1 - e^{-(T + \sqrt{-1}\theta)|\lambda_k^p|})]}. \quad (4.4)$$

*Proof.* The operator  $\mathcal{C}_{T^2}^{2,p}$  acting on  $\Omega^{0,*}(T_p M)$  splits  $S^1$ -equivariantly to  $n$  copies of

$$\mathcal{C}_{T^2}^2 = \frac{1}{2} \Delta + \frac{1}{2} T^2 |\lambda|^2 |z|^2 - \frac{1}{2} T \lambda (-1 + [d\bar{z} \wedge, i_{\partial/\partial \bar{z}}]), \quad \lambda \in \mathbb{Z} \setminus \{0\} \quad (4.5)$$

acting on  $\Omega^{0,*}(\mathbb{C})$ . Here  $S^1$  act on  $\mathbb{C}$  by  $R(\theta) = e^{\sqrt{-1}\lambda\theta}$  and hence on  $\Omega^{0,*}(\mathbb{C})$  as well. (4.5) is the sum of the Hamiltonian for the two dimensional harmonic oscillator

$$\mathcal{H}_{T^2}^2 = \frac{1}{2} \Delta + \frac{1}{2} T^2 |\lambda|^2 |z|^2 \quad (4.6)$$

and a bounded operator of order zero. The smooth heat kernel associated to the operator  $\exp(-\mathcal{H}_{T^2}^2)$  acting on  $\Omega^0(\mathbb{C})$  is given by Mehler's formula

$$K_{T^2}(z, z') = \frac{T|\lambda|}{2\pi \sinh T|\lambda|} \exp \left[ -T|\lambda| \left( \frac{|z|^2 + |z'|^2}{2 \tanh T|\lambda|} - \frac{\operatorname{Re}(\bar{z}z')}{\sinh T|\lambda|} \right) \right]. \quad (4.7)$$

Therefore

$$\begin{aligned} \operatorname{Tr}_{\Omega^0(\mathbb{C})}[R(\theta) \exp(-\mathcal{H}_{T^2}^2)] &= \int_{\mathbb{C}} d^2 z K_{T^2}(e^{-\sqrt{-1}\lambda\theta} z, z) \\ &= \frac{e^{-T|\lambda|}}{(1 - e^{-(T-\sqrt{-1}\theta)|\lambda|})(1 - e^{-(T+\sqrt{-1}\theta)|\lambda|})}. \end{aligned} \quad (4.8)$$

The bounded 0-th order operator in (4.5) takes values  $T\lambda$  and 0, respectively, on 0- and 1-forms. Furthermore, the  $S^1$ -action  $R(\theta)$  picks up a phase  $e^{\sqrt{-1}\lambda\theta}$  on  $d\bar{z}$ . Therefore

$$\operatorname{Tr}_{\Omega^{0,k}(\mathbb{C})}[R(\theta) \exp(-\mathcal{C}_{T^2}^2)] = \begin{cases} \frac{e^{-T|\lambda| - T\lambda}}{(1 - e^{-(T-\sqrt{-1}\theta)|\lambda|})(1 - e^{-(T+\sqrt{-1}\theta)|\lambda|})}, & \text{if } k = 0, \\ \frac{e^{-T|\lambda| + \sqrt{-1}\lambda\theta}}{(1 - e^{-(T-\sqrt{-1}\theta)|\lambda|})(1 - e^{-(T+\sqrt{-1}\theta)|\lambda|})}, & \text{if } k = 1. \end{cases} \quad (4.9)$$

Returning to the problem on  $T_p M$ , for  $I = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ , set  $d\bar{z}^I = d\bar{z}^{i_1} \wedge \dots \wedge d\bar{z}^{i_k}$ . Then we have

$$\operatorname{Tr}_{\Omega^0(T_p M) d\bar{z}^I}[R_p(\theta) \exp(-\mathcal{C}_{T^2}^2)] = \frac{e^{-T \sum_{k=0}^n |\lambda_k^p| - T \sum_{k \notin I} \lambda_k^p + \sqrt{-1}\theta \sum_{k \in I} \lambda_k^p}}{\prod_{k=1}^n [(1 - e^{-(T-\sqrt{-1}\theta)|\lambda_k^p|})(1 - e^{-(T+\sqrt{-1}\theta)|\lambda_k^p|})]}. \quad (4.10)$$

The trace on  $\Omega^{0,k}(T_p M)$  is the sum of (4.10) over  $I$  with  $|I| = k$ .  $\square$

(4.4) should be interpreted, after a Taylor expansion on the right hand side, as an equality of formal characters of  $S^1$ .

**Proof of formula (1.12):** In (4.1) we replace  $\theta$  formally by  $\theta + \sqrt{-1}T$  and still regard it as an equality of formal series in  $e^{\sqrt{-1}\theta}$ . Since as  $u \rightarrow 0$  the limit of the left hand side exists in  $\mathbb{R}((e^{\sqrt{-1}\theta}))$  (Corollary 3.6) and since  $H^k(\theta)$  is independent of  $u$ , we conclude that  $\lim_{u \rightarrow 0} Q_{u,T}(\theta + \sqrt{-1}T, t) = Q_T(\theta + \sqrt{-1}T, t)$  also exists and that  $Q_T(\theta, t) \geq 0$ . Therefore

$$\sum_{p \in F} E_p(\theta + \sqrt{-1}T) \operatorname{Tr}_{\Omega^{0,k}(T_p M)}[R_p(\theta) \exp(-\mathcal{C}_{T^2}^2)] = \sum_{k=0}^n t^k H^k(\theta + \sqrt{-1}T) + (1+t)Q_T(\theta + \sqrt{-1}T, t). \quad (4.11)$$

Using Lemma 4.2 and changing  $\theta + \sqrt{-1}T$  back to  $\theta$ , we get

$$\sum_{p \in F, I \subset \{1, \dots, n\}} t^{|I|} E_p(\theta) \frac{e^{-T \sum_{k=0}^n |\lambda_k^p| - T \sum_{k \notin I} \lambda_k^p + (T + \sqrt{-1}\theta) \sum_{k \in I} \lambda_k^p}}{\prod_{k=1}^n [(1 - e^{\sqrt{-1}|\lambda_k^p|\theta})(1 - e^{-|\lambda_k^p|(2T + \sqrt{-1}\theta)})]} = \sum_{k=0}^n t^k H^k(\theta) + (1+t)Q_T(\theta, t). \quad (4.12)$$

Finally, as  $T \rightarrow +\infty$ , the limit of each summand on the left hand side is 0 except when the pair  $(p, I)$  satisfies  $I = \{k \mid \lambda_k^p > 0\}$ , in which case the limit is  $t^{n-n_p} E_p(\theta) e^{\sqrt{-1} \sum_{\lambda_k^p > 0} |\lambda_k^p| \theta} / \prod_{k=1}^n (1 - e^{\sqrt{-1}|\lambda_k^p|\theta})$ . Consequently,  $\lim_{T \rightarrow +\infty} Q_T(\theta, t) = Q^-(\theta, t)$  exists as well, and  $Q^-(\theta, t) \geq 0$ . Formula (1.12) is proved.  $\square$

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